Lecture 23

The Borel Theory

k field

A graded k-algebra is a k-alg A and a vec space decomp

$$A = \bigoplus_i A_i \quad \text{w/ } k \subset A_0$$

s.t. $A_i A_j \subset A_{i+j}$. Elts in A_j have degree j.

e.g. $H^*(X, k)$.

$(V, \Phi, \langle , \rangle)$ root system w/ Weyl group W.

$\mathcal{P} = S(V)$ symm tensor algebra (graded alg $/\mathbb{R}$) $\mathcal{P}_0 = k$

$\mathcal{P}^W = W$-invt elts, also graded — $\cong k[x_1, \dots, x_r]$ invariants ring

$\mathcal{P}^W_+ \subset \mathcal{P}^W$ the elements of positive degree (elts of $\mathcal{P}^W$ w/ no k piece)

$J \subset \mathcal{P}^W$ the ideal gen by $\mathcal{P}^W_+$ (elts $\mathcal{P}$ divisible by an elt of $\mathcal{P}^W_+$)

$\mathcal{R} = \mathcal{P}/\mathcal{P}^W$ coinvariants ring

Relation to Wednesday: $\mathfrak{h} \cong \mathbb{C}^n = \text{Lie}((\mathbb{C}^*)^n)$ $\quad \mathfrak{h} = \underbrace{\mathfrak{a}}_{\text{Lie}((\mathbb{R}^+)^n)} \oplus \underbrace{\mathfrak{t}}_{\text{Lie}((S^1)^n)}$.

Root system has $V = \mathfrak{a}^*$

but the cplx str of $\mathfrak{h}$ gives a natural iso $\mathfrak{t} \cong \mathfrak{a}$ (mult by i)

So $V \cong \mathfrak{t}^*$. $\quad \mathcal{P} = S(\mathfrak{t}^*)$ as Wednesday.

Ex. $G = SL_n\mathbb{C}$ $\quad \mathfrak{t} =$ pure imag diag sum zero. $\begin{pmatrix} it_1 & & \\ & \ddots & \\ & & it_n \end{pmatrix}$

$x_k\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix} = t_k$. Then $\mathcal{P} \cong \mathbb{R}[x_1, \dots, x_n]/(x_1 + \dots + x_n)$ $(\cong \mathbb{R}[x_1 \dots x_{n-1}])$

$W = \text{Sym}_n$. $\quad \mathcal{P}^W =$ Symmetric poly in $x_1 \dots x_n / (x_1 + \dots + x_n)$

$$\cong \mathbb{R}[e_1, \dots, e_n]/(e_1) \cong \mathbb{R}[e_2, \dots, e_n]$$

$$e_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \qquad e_1 = x_1 + \cdots + x_n \qquad e_n = x_1 x_2 \cdots x_n$$

$$\mathcal{R} = P/(e_2, \ldots, e_n) = \mathbb{R}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$$

Key fact: generated in degree 1.

Explicit. $SL_3\mathbb{C}$. $\mathbb{R}[a,b,c]/(a+b+c,\ ab+bc+ca,\ abc)$

R_0 basis 1

R_1 basis a, b $\qquad (c = a+b-(a+b+c))$

R_2 basis ab, bc. (exercise: show $a^2 \in \mathbb{R}ab + \mathbb{R}bc + J$

R_3 basis abc

Thm (Chevalley-Shephard-Todd) Let W be a finite subgroup of $GL(V)$ generated by reflections in codim-1 subspaces.

Then 1) $S(V)^W$ is a poly ring w/ gens in different degrees $m_1 \ldots m_r$.

2) $S(V)$ is a free module over $S(V)^W$

There is a sort of converse, but refl needs to be replaced by pseudorefl.

In case of a Weyl group, $S(V)_1^W = V^W = \{0\}$ as W contains refl in a basis.

Thus $R_1 \cong V \cong \mathfrak{t}^*$.

Thm (Borel): There exists a natural degree-doubling $\mathbb{R}$-alg iso

$$\mathcal{R} \longrightarrow H^*(G/B, \mathbb{R})$$

In particular, H^* is generated as an alg by $H^2(G/B, \mathbb{R})$ which is isomorphic to $\mathfrak{t}^*$.

$$abc \longmapsto a \smile b \smile c$$

Note that there is a map $S(H^2(G/B, \mathbb{R})) \to H^*(G/B, \mathbb{R})$

Or in general a deg-pres map $S(A_1) \to A$.

So we could factor this into 3 statements:

1) There is an $\mathbb{R}$-vec sp iso $\mathfrak{t}^* \to H^2(G/B, \mathbb{R})$

2) H^2 generates H^*

3) $\ker(\mathrm{Sym}(H^2) \to H^*)$ "is" J.

Let's talk abt (1) first.

One way to construct an elt of H^2 is to take $[Z]$ where $Z \subset G/B$ is a complex submfld of codim 1.

Let $\lambda : \mathfrak{t} \to \mathbb{R}$ be a $\mathbb{R}$-linear map, i.e. $\lambda \in \mathfrak{t}^*$.

Extend it to a $\mathbb{C}$-linear map $\lambda : \mathfrak{h} \to \mathbb{C}$.

Then $\tilde{\lambda} = \exp(i\lambda) : H \to \mathbb{C}^*$ grp hom.

There is a retraction $B \to H$ taking $b = hu$ to h.

So we get $\tilde{\lambda} : B \to \mathbb{C}$.] actually requires integrality!

A function on G/B is a right-B-invt map $G \to \mathbb{C}$.

— holo fn ———————— holo map —

<u>Note</u>. the only holo fns on G/B are const. (But lots of <u>mero</u>.)

<u>Def</u>. A λ-twisted fn on G/B is a map $s : G \to \mathbb{C}$ that is holo and s.t.
$$s(gb) = \underbrace{\tilde{\lambda}(b^{-1})}_{\mathbb{C}^*} s(g) \quad \forall\, b \in B.$$

This actually means: View G as B-bundle over G/B.

Then use $\tilde{\lambda}$ to replace each fiber w/ $\mathbb{C}$ to get line bundle.

<u>The map</u>. Prelim

$c: \lambda \in \mathfrak{t}^* \longmapsto [Z]$ where $Z = \left(f^{-1}(0)\right)/B$ where $f: G \to \mathbb{C}$ λ-twisted.

<u>Actual</u>. Let $F: G \to \mathbb{C}$ be a λ-twisted meromorphic map. Let $\sum_i n_i Z_i$ be the divisor in G/B that is the image of $\mathrm{div}(F)$, where $n_i \in \mathbb{Z}$, $Z_i \subset G/B$ irred codim-1 proj subvars.

<u>Or</u>. Let $\mathcal{L}_\lambda = (G \times_{\tilde{\lambda}} \mathbb{C})/B$. Then $c(\lambda) = -c_1(\mathcal{L}_\lambda)$.

<u>Why is this an isomorphism?</u> Given elt $H^2(G/B, \mathbb{R})$, how to realize it?